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Noise Sources in Turbulent Shear Flows: Fundamentals and Applications

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PREFACE

The knowledge of the physical mechanisms underlying the generation of noise in turbulent shear flows remains a challenging task despite over 50 years of intensive research in the field. The interest in this topic is considerable because turbulent shear flows originating noise are encountered in many engineering applications, such as flows in pipes, compressible and incompressible jets, turbulent boundary layers over rigid or elastic surfaces, wakes generated behind streamlined or bluff bodies.

Recent developments in terms of our capacity to both numerically and experimentally analyze the physics of turbulent shear flows have opened up new possibilities to improve our knowledge about noise generation and propagation mechanisms. These understandings lead, for example, to the development of flow/noise manipulation techniques and address the design of noise suppression devices.

The scope of this volume is to present a state-of-the-art review of on-going activities in noise prediction, modeling and measurement and to indicate current research directions. This book is partially based on class notes provided during the course 'Noise sources in turbulent shear flows', given at CISM on April 2011.

Introductory chapters on fundamentals topics will be followed by up-to-date reviews of arguments of specific interest for engineering applications.

The first part of the volume is denoted as 'Fundamentals' and contains two chapters. The first one covers general concepts of aeroacoustics, from the basic equations of fluid dynamics to the theoretical description of self-sustained oscillations in internal flows including the vortex sound theory. The second chapter illustrates more deeply the acoustic analogies in account also of the presence of solid surfaces. The flow features involved in sound generation are also highlighted by means of suitable dimensional analyses.

In the second part of the volume, denoted as 'Applications', particular emphasis is put into arguments of interest for engineers and relevant for aircraft design. An important topic included in this part is jet noise, which is treated from both an experimental and an analytical viewpoint. A comprehensive review of literature results as well *as a description of present understandings of noise generation and its predictions is presented.*

A second chapter is devoted to describing airfoil broadband noise and its analytical modeling with emphasis on trailing edge noise and rotating blades.

The boundary layer noise is treated in another chapter that is divided into two parts. In the first one noise generation mechanisms are described. In the second, the problem of the interior noise and some basic approaches used for its control are presented.

As a fundamental completion of the state-of-the-art knowledge, a chapter is devoted to clarifying the concept of noise sources, their theoretical modeling and the techniques used for their identification in turbulent flows.

All these arguments are treated extensively with the inclusion of many practical examples and references to engineering applications.

For the purpose of optimizing the convenience of this book, the chapters are conceived to be self-contained. Readers may concentrate on the topic they are more interested in, with no need of consulting other chapters. The disadvantage of this approach lies in the repetition of some basic notions, such as the Lighthill's analogy or the Green's function formalism, which can be found replicated in more than one chapter. Indeed, scientists may use the same mathematical tool in a different but efficient way, depending on the purpose of their analysis.

To my opinion, these reiterations do not represent a shortcoming. On the contrary I consider this approach to be a quite instructive way for young researchers to discover and appreciate the amazing strength and effectiveness of theories, models and mathematical formalisms that provide the foundations of aeroacoustics.

Roberto Camussi

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Part 1: Fundamentals

Introduction to Aeroacoustics and Self-Sustained Oscillations of Internal Flows

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Abstract After a review of basic equations of fluid dynamics, the *Aeroacoustic analogy* of Lighthill is derived. This analogy describes the sound field generated by a complex flow from the point of view of a listener immerged in a uniform stagnant fluid. The concept of monopole, dipole and quadrupole are introduced. The scaling of the sound power generated by a subsonic free jet is explained, providing an example of the use of the integral formulation of the analogy. The influence of the Doppler Effect on the radiation of sound by a moving source is explained. By considering the noise generated by a free jet in a bubbly liquid, we illustrate the importance of the choice of the aeroacoustic variable in an aeroacoustic analogy. This provides some insight into the usefulness of alternative formulations, such as the *Vortex Sound Theory.* The energy corrolary of Howe based on the Vortex Sound Theory appears to be the most suitable theory to understand various aspects of self-sustained oscillation due to the coupling of vortex shedding with acoustic standing waves in a resonator. This approach is used to analyse the convective energy losses at an open pipe termination, human whistling, flow instabilities in diffusers, pulsations in pipe systems with deep closed side branches and the whistling of corrugated pipes.

1 Introduction

Due to the essential non-linearity of the governing equations it is difficult to predict accurately fluid flows under conditions at which they do produce sound. This is typical for high speeds with non-linear inertial terms in the equation of motion much larger than the viscous terms (high Reynolds numbers). Direct simulation of such flows is very difficult. When the flow velocity remains low compared to the speed of sound waves (low Mach numbers) the sound production is a minute fraction of the energy in the flow,

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making numerical simulation even more difficult. It is not even obvious how one should define the acoustic field in the presence of flows. Aeroacoustics does provide such definitions. The acoustic field is defined as an extrapolation of an ideal reference flow. The difference between the actual flow and this reference flow is identified as source of sound. Using Lighthill's terminology, we call this an "analogy" [Lighthill (1952-54)].

In free field conditions the sound intensity produced by flows is usually so small that we can neglect the effect of acoustics on the flow. Furthermore, the listener is usually immerged in a uniform stagnant fluid. In such cases the convenient reference flow is the linear inviscid perturbation of this stagnant, uniform fluid. It is convenient to use an integral formulation of the aero-acoustical analogy. This integral equation is a convolution of the sound source by the Green function: the response of the reference state to a localized impulsive source. The advantage of the integral formulation is that random errors in the source term average out. One therefore often uses such an integral formulation to extract acoustic information from direct numerical simulations of the flow which are too rough to directly predict the acoustic field. Such an approach is used so as to obtain scaling laws for sound production by turbulent flows when only global information is available about the flow. When flow dimensions are small compared to the acoustical wave length (compact flow) we can locally neglect the effect of wave propagation within the source region. Here the analogy of Lighthill provides again a procedure which guarantees that we keep the leading order term where brute force would predict no sound production at all or would dramatically overestimate it [Crighton et al. (1992)]. In compact flows at low Mach numbers the flow is most efficiently described in terms of vortex dynamics, allowing a more detailed study of the sound production by nonlinear convective effects.

Walls have a dramatic effect on the production of sound because it becomes much easier compressing the fluid than in free space. In internal flows acoustic energy can accumulate into standing waves, which correspond to resonances. Even at low Mach numbers acoustical particle velocities of the order of magnitude of the main flow velocity can be reached when hydrodynamic flow instabilities couple with the acoustic standing waves. This relatively high amplitude facilitates numerical simulations considerably. Such self-sustained oscillations are best described qualitatively in terms of vortex dynamics.

In a pipe the main flow does not necessarily vanish when travelling away from the source region. For these reasons another analogy should be used, called the Vortex-Sound Theory. Whilst Powell (1964) initially developed this theory for free space, Howe generalised it for internal flows [Howe (1975), Howe (1984), Howe (1998), Howe (2002)]. In Howe's approach the acoustic field is defined as the unsteady irrotational component of the flow, which again stresses the fact that vortices are the main sources of sound in isentropic flows. An integral formulation can also be used in this case.

When considering self-sustained oscillations, one is interested in conditions at which they appear and the amplitude they reach. While a linear theory provides information on the conditions under which self-sustained oscillation appears, the amplitude is determined by essentially non-linear saturation mechanisms. We will show that when ever the relevant non-linear mechanism is identified, the order of magnitude of steady self-sustained pulsation amplitude can be easily obtained. A balance between the acoustic power produced by the source and the dissipated power will be used.

A summary of the equations of fluid dynamics is given in (section 2). In Section 3 we introduce the acoustic field by means of Lighthill's analogy, followed by basic concepts of the acoustics of a stagnant uniform fluid, such as elementary solutions of the wave equation, acoustic energy, the Green function, multipole expansion, Doppler effect and convective effects due to a uniform main flow (section 4). We use the analogy of Lighthill to derive the scaling law for sound production by a subsonic isothermal free jet. The influence of the difference in speed of sound between the source region and the listener is discussed by using the example of bubbly liquids (section 5). We then introduce the acoustics of pipes, derive the low frequency limit of acoustic properties of a pipe discontinuity and of an open pipe termination (with and without main flow). In Section 6 we introduce the concepts of resonators and discuss closed-side branch and Helmholtz resonators. In section 7 we introduce vortex sound theory and apply it to the analysis of whistling, from human whistling to whistling of corrugated pipes. Some aspects introduced here are discussed in depth in the following chapters.

Our discussion is inspired by the book of Dowling and Ffowcs Williams (1983), which is an excellent introductory course. Basic acoustics is discussed in the books of Morse and Ingard (1968), Pierce (1990), Kinsler et al. (1982), Temkin (2001), Blackstock (2000) and Bruneau (2006). Aeroacoustics is treated in the books of Goldstein (1976), Blake (1986), Crighton et al. (1992), Howe (1998)and Howe (2002). In this introduction we ignore the effect of wall vibration [Junger and Feit (1986), Cremer and Heckl (1988) and Norton (1989)]. Acoustics of musical instruments is discussed by Fletcher and Rossing (1998) and Chaigne and Kergomard (2008). In an earlier course Hirschberg et al. (1995) and a review paper [Fabre et al. (2012)] we discussed the aeroacoustics of woodwinds. In the Lecture notes of Rienstra and Hirschberg (1999) provide more details on the mathematical aspects.

2 Fluid dynamics

2.1 Conservation laws

The conservation of mass for an infinitesimal material element of density ρ and volume V is given in the continuum approximation by [Batchelor (1967), Landau and Lifchitz (1987), Kundu (1990)]:

$$\frac{D\rho V}{Dt} = 0 \tag{1}$$

where the convective time derivative is defined by:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{v} \cdot \nabla)\,\rho \tag{2}$$

in vector notation. In the index notation we have:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i}\ . \tag{3}$$

Following the convention of Einstein, a summation is assumed in equation (3) over the repeated index $i = 1, 2, 3$. The dilation rate of a fluid particle is given by:

$$\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \vec{v} = \frac{\partial v_i}{\partial x_i} \tag{4}$$

Hence, the mass conservation law (1) can be written in the conservation form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0\ . \tag{5}$$

In integral form this equation becomes:

$$\frac{d}{dt}\int_V \rho dV + \int_S \rho\,(\vec{v} \cdot \vec{n})\,dS = 0 \tag{6}$$

in which, V is a fixed control volume delimited by the surface S with outer unit normal $\vec{n}$ (Figure 1). The second law of Newton applied to an infinites-

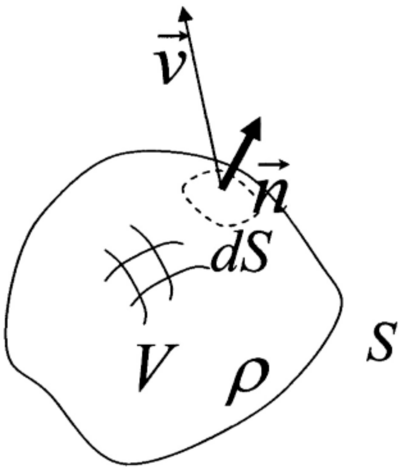

Figure 1. Control volume used to establish the integral conservation laws.

imal material element is:

$$\rho \frac{D\vec{v}}{Dt} = -\nabla \cdot \vec{\vec{P}} + \vec{f} \tag{7}$$

where $\vec{f}$ is the density of a force field acting on the bulk of the fluid and $\vec{\vec{P}}$ is the stress tensor representing the surface interaction between the particle and its surroundings. Using the definition of the convective derivative (2) and the mass conservation law (5) we obtain the conservation form of the momentum equation:

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla \cdot \vec{\vec{P}} + \vec{f} \tag{8}$$

or in index notation:

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial P_{ij}}{\partial x_j} + f_i \tag{9}$$

and integral form:

$$\frac{d}{dt}\int_V \rho \vec{v} dV + \int_S \rho \vec{v} \left(\vec{v} \cdot \vec{n}\right) dS = -\int_S \vec{\vec{P}} \cdot \vec{n} dS + \int_V \vec{f} dV \ . \tag{10}$$

The energy conservation law is, in differential form:

$$\frac{D\rho(e + v^2/2)}{Dt} = -\nabla \cdot \vec{q} - \nabla \cdot \left(\vec{\vec{P}} \cdot \vec{v}\right) + \vec{f} \cdot \vec{v} + Q_w \tag{11}$$

where e is the internal energy of the fluid per unit of mass, $v = |\vec{v}|$, $\vec{q}$ the heat flux and Q_w the energy production per unit volume.

2.2 Constitutive equations

The conservation laws are complemented by empirical constitutive equations. For simplicity we assume that the fluid is locally in a state close to thermodynamic equilibrium, so that we can express the internal energy in terms of two other state variables:

$$e = e(\rho, s) \tag{12}$$

where s is the entropy per unit of mass. Using the thermodynamic equation:

$$de = Tds - pd\left(\frac{1}{\rho}\right) \tag{13}$$

we get the equations of state:

$$p = \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_s \tag{14}$$

and

$$T = \left(\frac{\partial e}{\partial s}\right)_\rho \ . \tag{15}$$

As we also have $p = p(\rho, s)$ we can write:

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds \ . \tag{16}$$

The speed of sound c is defined by:

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \ . \tag{17}$$

In most applications we will consider an ideal gas for which:

$$de = c_v dT \tag{18}$$

with c_v the specific heat capacity at constant volume. For an ideal gas this is a function of the temperature only. This further implies:

$$p = \rho R T \tag{19}$$

and

$$c = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}} \tag{20}$$

with $R = c_p - c_v$ the specific gas constant, $\gamma = c_p/c_v$ the Poisson ratio and c_p is the specific heat capacity at constant pressure. By definition:

$$c_v = \left(\frac{\partial e}{\partial T}\right)_\rho \tag{21}$$

and

$$c_p = \left(\frac{\partial i}{\partial T}\right)_p \tag{22}$$

where the specific enthalpy is defined by:

$$i = e + \frac{p}{\rho} \ . \tag{23}$$

Assuming local thermodynamic equilibrium, fluxes are linear functions of the flow variables. For the heat flux we use the law of Fourier:

$$\vec{q} = -K\nabla T \ , \tag{24}$$

where K is the heat conductivity. The viscous stress tensor is defined by:

$$\tau_{ij} = p\delta_{ij} - P_{ij} \tag{25}$$

with δ_{ij} the Kronecker delta, equal to unity for $i = j$ and otherwise zero. The viscous stress tensor is described for a so-called Newtonian fluid in terms of the dynamic viscosity η and the bulk viscosity μ:

$$\tau_{ij} = 2\eta \left(D_{ij} - \tfrac{1}{3} D_{kk}\delta_{ij}\right) + \mu D_{kk}\delta_{ij} \tag{26}$$

with

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \ . \tag{27}$$

2.3 Boundary conditions

The boundary conditions corresponding to the continuum assumption and the local thermodynamic equilibrium are, for a solid impermeable wall with velocity $\vec{v}_w$ and temperature $T_w : \vec{v} = \vec{v}_w$ and $T = T_w$.

2.4 Approximations

Sound production by flows occurs at relatively high Reynolds numbers. When considering wave propagation in air at audio frequencies, we can neglect friction and heat transfer over distances of the order of the wave length. Neglecting friction, heat transfer and heat production, the energy equation (11) becomes:

$$\frac{Ds}{Dt} = 0 \ . \tag{28}$$

The momentum equation (7) reduces to the Euler equation:

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \vec{f} \ . \tag{29}$$

In terms of the vorticity $\vec{\omega} = \nabla \times \vec{v}$ the convective acceleration can be written as:

$$(\vec{v} \cdot \nabla)\, \vec{v} = \nabla \left(\frac{v^2}{2} \right) + \vec{\omega} \times \vec{v} \ . \tag{30}$$

For homentropic flows $\nabla s = 0$ we have furthermore $\nabla p/\rho = \nabla i$, so that the Euler equation can be written in the form of Crocco:

$$\frac{\partial \vec{v}}{\partial t} + \nabla B = -\left(\vec{\omega} \times \vec{v} \right) + \frac{\vec{f}}{\rho} \tag{31}$$

with the total enthalpy:

$$B = i + v^2/2 \ . \tag{32}$$

For irrotational flow $\vec{\omega} = 0$ we can introduce a velocity potential such that:

$$\vec{v} = \nabla \varphi \tag{33}$$

or:

$$\varphi = \int \vec{v} \cdot d\vec{x} \ . \tag{34}$$

In the absence of an external force field, the integration of the Euler equation yields the unsteady compressible Bernoulli equation:

$$\frac{\partial \varphi}{\partial t} + B = g(t) \tag{35}$$

in which the function $g(t)$ is determined by the boundary conditions.

In this isentropic flow approximation $dQ = 0$, so that it follows from the first law of thermodynamics:

$$dQ = de + pd\left(\frac{1}{\rho}\right) = di - \frac{1}{\rho}dp \tag{36}$$

and

$$i = \int \frac{dp}{\rho} \ . \tag{37}$$

3 Analogy of Lighthill

The key idea of Lighthill's analogy [Lighthill (1952-54)] is to derive a wave equation starting from the exact mass conservation equation (5) and the momentum equation (9):

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_i}{\partial x_i} = 0 \tag{5}$$

and

$$\frac{\partial\rho v_i}{\partial t} + \frac{\partial\rho v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ij}}{\partial x_j} + f_i \ . \tag{9}$$

Taking the time derivative of (5) and subtracting from it, the divergence of (9) we obtain the exact equation:

$$\frac{\partial^2\rho}{\partial t^2} - \frac{\partial^2\rho v_i v_j}{\partial x_i\partial x_j} = \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2\tau_{ij}}{\partial x_i\partial x_j} - \frac{\partial f_i}{\partial x_i} \tag{38}$$

which is quite meaningless. By adding $\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$ on both sides and rearranging the terms, making use of the fact that we chose c_0 to be a constant, we can write (38) as a wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial^2 \rho v_i v_j - \tau_{ij}}{\partial x_i\partial x_j} - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial t^2}\left(\frac{p}{c_0^2} - \rho\right) \ . \tag{39}$$

This equation is still exact and still generally meaningless. We could have chosen c_0 to be a millimetre per century or equal to the speed of light. In order to have a meaningful equation we now assume that we consider sound production by a flow bounded by a fluid displaying small perturbations from a uniform stagnant state with speed of sound equal to c_0 (Figure 2). We furthermore define the perturbations in the pressure $p' = p - p_0$ and density $\rho' = \rho - \rho_0$ as deviations from the state (p_0, ρ_0) of this reference uniform

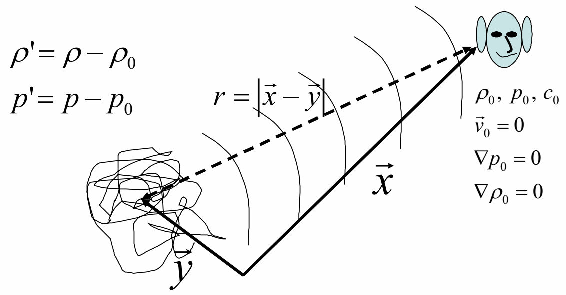

Figure 2. Sound sources and listener in the analogy of Lighthill

stagnant reference fluid. As the reference state is constant and uniform we can write (39) as:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2 \rho v_i v_j - \tau_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial t^2}\left(\frac{p'}{c_0^2} - \rho'\right) \ . \tag{40}$$

We will see (section 4) that this equation describes the propagation of acoustic waves in the uniform stagnant fluid when the right hand side of the equation (40) is negligible. In regions where the right hand side is not negligible, it describes the generation of sound. However, because the equation of Lighthill is a single exact equation for many unknowns, we will not obtain any result without approximations. Lighthill has shown that these approximations can best be introduced into an integral formulation of (40). We will now consider basic acoustic wave propagation allowing to understand some elementary aspects of the problem and to derive the integral formulation.

An interesting aspect of the analogy is that the sound source we find depends on the choice of the acoustic variable. Until now we have chosen pressure fluctuations p' to describe the acoustic field. We could also have followed a similar procedure to obtain a wave equation for the density fluctuations ρ'. Starting from (38) we now subtract from both sides of the equation the term $c_0^2 \nabla^2 \rho'$ to find:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 \rho v_i v_j - \tau_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial x_i^2}\left(p' - c_0^2 \rho'\right) \ . \tag{41}$$

In principle equations (40) and (41) are identical. However the pressure formulation (40) is most convenient when considering sound production by combustion processes in which the time-dependent combustion yields time-dependent fluctuations in the entropy. In contrast, when considering a flow in spatially non-uniform fluids with large variations in speed of sound the density formulation (41) will be the most suitable. An example of this is the sound generation by turbulence in bubbly liquids (section 5.2). In this case the sound production appears to be dominated by the effect of differences in the speed of sound.

Equation (41) is often written for convenience in terms of the stress tensor of Lighthill :

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i}{\partial x_i} \tag{42}$$

where the stress tensor τ_{ij} is defined by:

$$T_{ij} = \rho v_i v_j - \tau_{ij} + \left(p' - c_0^2 \rho'\right) \delta_{ij} \ . \tag{43}$$

4 Acoustics of a uniform stagnant fluid

4.1 Wave equation

Looking at small perturbations $(p', \rho', \vec{v}')$ of a uniform stagnant state (p_0, ρ_0) and neglecting friction and heat transfer, we find, for linear perturbations:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v}' = 0 \ , \qquad (44)$$

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} + \nabla p' = \vec{f} \qquad (45)$$

and

$$\frac{\partial s'}{\partial t} = \frac{Q_w}{\rho_0 T_0} \ . \qquad (46)$$

The corresponding linearized equation of state is:

$$p' = c_0^2 \rho' + \left(\frac{\partial p}{\partial s} \right)_\rho s' \ . \qquad (47)$$

Taking the time derivative of (44), subtracting the divergence of (45) and using (46) and (47) in order to eliminate ρ' and s', we obtain the wave equation for pressure perturbations:

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{1}{T_0 \rho_0 c_0^2} \left(\frac{\partial p}{\partial s} \right)_\rho \frac{\partial Q_w}{\partial t} - \nabla \cdot \vec{f} \ . \qquad (48)$$

As can be seen from this equation, the unsteady heat production is a source of sound, which is due to the dilatation of the fluid. This is in line with our common experience that turbulent flames are noisy. Also an unsteady non-uniform force field appears to be a source of sound. This is the sound source when considering the whistling of a cylinder placed with its axis normal to a uniform flow. Due to hydrodynamic instability, the wave behind the cylinder breaks down into a vortex street of alternating rotation direction. This periodic vortex shedding induces an unsteady force of the flow on the cylinder. The reaction force from the cylinder on the fluid is the source of sound. The so-called Aelonian tone will be discussed in section 7.2.

The next sections will focus on wave propagation and hence assume that $Q_w = 0$ and $\vec{f} = 0$. We therefore consider solutions of the homogeneous wave equation of d'Alembert

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \ . \qquad (49)$$

As the flow is isentropic the equation of state (16) reduces to $p' = c_0^2 \rho'$.

4.2 Elementary solutions

The homogeneous scalar wave equation (49) satisfies the plane wave solution:

$$p' = F\left(\vec{n}\cdot\vec{x} - c_0 t\right) \tag{50}$$

with $\vec{n}$ as the unit vector in the direction of propagation. This can easily be verified for $\vec{n} = (1,0,0)$, in which case the wave equation (45) reduces to:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0\ . \tag{51}$$

Using the chain rule we can verify that $p' = F(x - c_0 t)$ is a solution. The function $F(x)$ is determined by initial and boundary conditions. Also $p' = G(x + c_0 t)$ is a solution, representing a wave propagating in the opposite direction $\vec{n} = (-1,0,0)$. For harmonic waves with a frequency f we can write this solution with the complex notation as:

$$p' = A\exp\left[i\omega\left(t - \frac{\vec{n}\cdot\vec{x}}{c_0}\right)\right] = A\exp\left[i\left(\omega t - \vec{k}\cdot\vec{x}\right)\right] \tag{52}$$

where A is the complex amplitude, $\vec{k} = (\omega/c_0)\vec{n}$ the wave vector and $\omega = 2\pi f$. Substitution of the plane wave solution into the momentum equation (45)with $\vec{f} = 0$ yields:

$$\vec{u}' = \frac{p'}{\rho_0 c_0}\vec{n}\ . \tag{53}$$

Another elementary solution is obtained by considering spherical symmetric waves emanating from a point at source $\vec{y}$. The pressure field is then only a function of time and of distance $r = |\vec{x} - \vec{y}|$ between the source position $\vec{y}$ and the observer's position $\vec{x}$. The mass conservation law and momentum equation reduce to:

$$\frac{\partial \rho'}{\partial t} + \frac{\rho_0}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r'}{\partial r}\right) = 0 \tag{54}$$

and

$$\rho_0\frac{\partial v_r'}{\partial t} + \frac{\partial p'}{\partial r} = 0 \tag{55}$$

where v_r' is the fluid velocity in the radial direction. Eliminating the velocity and the density $\rho' = p'/c_0^2$ yields:

$$\frac{1}{c_0^2}\frac{\partial^2 rp'}{\partial t^2} - \frac{\partial^2 rp'}{\partial r^2} = 0 \tag{56}$$

which is satisfied by the one-dimensional d'Alembert solution for the product of pressure p' and distance r:

$$p' = \frac{1}{r} F(r - c_0 t) \; . \tag{57}$$

By using this equation, we actually assume "free field" conditions. We assume that there are only outgoing waves and no incoming (or reflected) waves converging towards the source. For harmonic waves equation (57) becomes in complex notation:

$$p' = \frac{A}{r} \exp\left[i(\omega t - kr)\right] \tag{58}$$

with $k = \omega / c_0$. The corresponding radial velocity is found by substitution in the momentum equation:

$$v'_r = \frac{p'}{\rho_0 c_0} \left[1 + \frac{1}{ikr}\right] \; . \tag{59}$$

We observe that for large distances compared to the wave length $kr >> 1$, the solution can locally be approximated by a plane wave with: $p' = \rho_0 c_0 v'_r$. In this so-called "far field" approximation we have:

$$\frac{\partial p'}{\partial r} \approx -\frac{1}{c_0} \frac{\partial p'}{\partial t} \; . \tag{60}$$

In the opposite limit of near field $kr << 1$ the velocity varies quadratically with the distance r, which is typical for the incompressible flow from a point volume source. Whenever characteristic flow dimensions are small compared to the wave length we can neglect wave propagation. Such a flow is called a "compact" flow.

Using these results (58-59) we can now consider the sound radiated by a pulsating sphere of radius

$$a = a_0 + \widehat{a} \exp(i\omega t) \tag{61}$$

where $\widehat{a}/a_0 << 1$ and $\omega \widehat{a}/c_0 << 1$. Substituting (61) into (59) and using (58) we find:

$$i\omega \widehat{a} = \frac{A \exp(-ika_0)}{\rho_0 c_0 a_0} \left(1 + \frac{1}{ika_0}\right) \tag{62}$$

and

$$p' = -\frac{\rho_0 \omega^2 a_0 \widehat{a}}{(1 + (ka_0)^2)} (1 - ika_0) \left(\frac{a_0}{r}\right) \exp\left[i(\omega t - k(r - a_0))\right] \; . \tag{63}$$

This result shows that in the limit $ka_0 << 1$ for a given volume flux amplitude $4\pi a_0^2 \omega \widehat{a}$, the amplitude of the radiated sound wave increases linearly with the frequency. At low frequency the pulsating sphere is compact and is a very inefficient source of sound. In the opposite limit $ka_0 >> 1$ the radiated amplitude is independent of the frequency.

4.3 Acoustic energy

For further reference we now consider the acoustic energy. Following the original approach of Kirchhoff, we start from the linearized mass and momentum equations:

$$\frac{1}{c_0^2}\frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot \vec{v}' = \frac{1}{T_0 \rho_0 c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho Q_w \tag{64}$$

and

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} + \nabla p' = \vec{f}\ . \tag{65}$$

Then we multiply the mass conservation law by p'/ρ_0 and add the in-product of the momentum equation with the velocity $\vec{v}'$, to find:

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{I} = \frac{1}{(\rho_0 c_0)^2 T_0}\left(\frac{\partial p}{\partial s}\right)_\rho p' Q_w + \vec{v}' \cdot \vec{f} \tag{66}$$

with the acoustic energy density E defined by :

$$E = \frac{1}{2}\rho_0 |\vec{v}'|^2 + \frac{(p')^2}{2\rho_0 c_0^2} \tag{67}$$

and the intensity I defined by:

$$\vec{I} = p'\vec{v}'\ . \tag{68}$$

It should be noted that this derivation assumes that we did not neglect any relevant quadratic terms when using the linear approximation for the mass and momentum equation. This approach appears to be valid only for the case considered, i.e. of a uniform stagnant reference state [Morfey (1971), Landau and Lifchitz (1987), Pierce (1990), Myers (1991)].

Equation (66) clearly shows generating acoustic energy requires that a volume source should be placed at a position with a large acoustic pressure. A force needs an acoustic velocity to generate acoustic energy.

Considering a compact pulsating sphere near a rigid plane wave $kh << 1$ (Figure 3), we observe that due to reflection at the wall the amplitude of waves reaching an observer in the far field is roughly double the amplitude we would find in free space. Hence, the intensity is four times larger than in free space. However, the source only radiates into a half space, so that the time averaged power $< P >$ generated by the source is doubled. This result can also be understood as a result of the doubling of the pressure fluctuations surrounding the source, due to reflection at the wall, which, following our energy corollary doubles the generated power. This implies

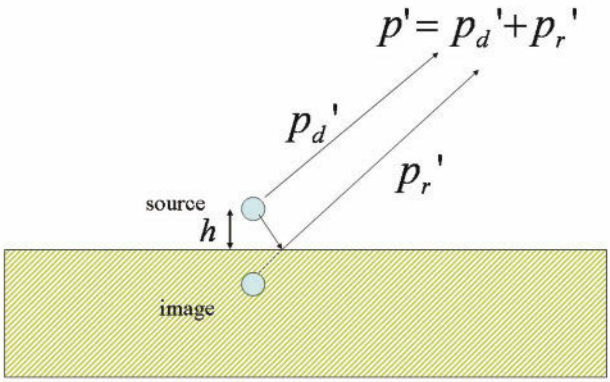

Figure 3. Influence of a rigid plane wall on the radiation of a compact sphere placed near the wall: $p' = p'_d + p'_r \approx 2p'_d \Rightarrow < P >= \frac{1}{2} 4\pi r^2 < I_r >= 2 \times 4\pi r^2 < \frac{(p'_d)^2}{\rho_0 c_0} >$

that the radiated power is doubled compared to free field conditions. This example stresses the fact that the sound power does not only depend on the source but also on the surroundings of the source.

4.4 Free space Green's function and integral formulation

Using the superposition principle we obtain an integral formulation of the wave equation for free space conditions. We first consider the sound generated by a pulse from a point source. This implies a localization in time and space, obtained by using the delta function. The delta function $\delta(t)$ is a generalized function defined by [Chrighton (1992)]:

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0) \ . \tag{69}$$

For any well behaving function $f(t)$ and:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \ . \tag{70}$$

The delta function has no meaning outside an integral. The free-field Green function $G_0(\vec{x}, t|\vec{y}, \tau)$ is the solution of the wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 G_0}{\partial t^2} - \nabla^2 G_0 = \delta(t-\tau)\delta(\vec{x}-\vec{y}) \tag{71}$$

where $\delta(\vec{x}-\vec{y}) = \delta(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3)$, for free-field boundary conditions and for the initial conditions:

$$G_0(\vec{x}, t|\vec{y}, \tau) = 0, \quad t \leq \tau \tag{72}$$

and

$$\frac{\partial}{\partial t} G_0(\vec{x}, t|\vec{y}, \tau) = 0, \quad t \leq \tau \tag{73}$$

corresponding to the causality condition that a wave cannot reach an observer before it has been emitted. In order to determine $G_0(\vec{x}, t|\vec{y}, \tau)$ we use the Fourier transform $\widehat{G}_0$ defined by:

$$G_0(\vec{x}, t|\vec{y}, \tau) = \int_{-\infty}^{\infty} \widehat{G}_0(\omega, \vec{x}|\vec{y}) \exp(i\omega t) d\omega \tag{74}$$

and

$$\widehat{G}_0(\omega, \vec{x}|\vec{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_0(\vec{x}, t|\vec{y}, \tau) \exp(-i\omega t) dt \ . \tag{75}$$

As we consider the field generated by a point source in free-field conditions we know that the Fourier transform of the Green function is given by:

$$\widehat{G}_0(\omega, \vec{x}|\vec{y}) = \frac{A}{r}\exp(-ikr) \tag{76}$$

where A is an amplitude which will be determined by using the properties of the delta function. We take the Fourier transform of the wave equation (71). Using the property (69) of the delta function:

$$\begin{aligned} &\tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}\delta(t-\tau)\exp(-i\omega t)dt = \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty-\tau}^{\infty-\tau}\delta(t-\tau)\exp(-i\omega(t-\tau)-i\omega\tau)d(t-\tau) = \tfrac{\exp(-i\omega\tau)}{2\pi} \end{aligned} \tag{77}$$

we find:

$$-(k^2+\nabla^2)\widehat{G}_0 = \frac{\exp(-i\omega\tau)}{2\pi} \ . \tag{78}$$

We integrate this equation over a spherical volume V of radius R enclosing the source:

$$-\int_V (k^2+\nabla^2)\widehat{G}_0 dV = \frac{\exp(-i\omega\tau)}{2\pi} \ . \tag{79}$$

By taking the limit of a compact control volume $kR << 1$ and using the Gaussian Theorem we find:

$$-\int_s \frac{\partial \widehat{G}_0}{\partial r} dS = -4\pi R^2 \left.\frac{\partial \widehat{G}_0}{\partial r}\right|_{r=R} = 4\pi R^2 \frac{A}{R^2} = \frac{\exp(-i\omega\tau)}{2\pi} \tag{80}$$

which yields the amplitude A. Substituting A in (76) and transforming back to the time domain yields:

$$G_0(\vec{x},t|\vec{y},\tau) = \frac{\delta(\tau - t_e)}{4\pi r} \tag{81}$$

where the emission (retarded) time t_e is defined by:

$$t_e = t - \frac{r}{c_0} \ . \tag{82}$$

Because Green's function in free-space only depends on the distance r and time difference $(t-\tau)$, rather than on the source and observer's coordinates $(\vec{x},t)$ and $(\vec{y},\tau)$ separately, it satisfies the important symmetry properties:

$$G_0(\vec{x},t|\vec{y},\tau) = G_0(\vec{y},-\tau|\vec{x},-t) \tag{83}$$

and

$$\frac{\partial G_0}{\partial t} = -\frac{\partial G_0}{\partial \tau} \tag{84}$$

and

$$\frac{\partial G_0}{\partial x_i} = -\frac{\partial G_0}{\partial y_i} \ . \tag{85}$$

Equation (83) is the so-called reciprocity relation, which is also valid for Green's functions in the presence of walls.

We can now use the Green function to build the free-field solution of the non-homogeneous wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = q(\vec{x},t) \tag{86}$$

by using the superposition principle:

$$p'(\vec{x},t) = \int_{-\infty}^{t}\int_V q(\vec{y},\tau)G_0(\vec{x},t|\vec{y},\tau)dV_y d\tau = \int_V \frac{q(\vec{y},t_e)}{4\pi r}dV_y \tag{87}$$

where $dV_y = dy_1 dy_2 dy_3$.

Substitution of (87) into (86) and using the definition (71) of Green's function we can verify the validity of this solution.

In the presence of walls, we can still use the same free-field Green function. However, now the solution of the wave equation will include surface integrals representing the effect of reflections of waves at the walls. Using Green's theorem we have:

$$\begin{aligned} p'(\vec{x},t) = \;& \int_{-\infty}^{t}\int_V q(\vec{y},\tau)G_0(\vec{x},t|\vec{y},\tau)dV_y d\tau \\ & -\int_{-\infty}^{t}\int_S \left[p'\nabla_y G_0 - G_0\nabla_y p'\right]\cdot\vec{n}dS_y d\tau \;. \end{aligned} \tag{88}$$

This integral formulation, in combination with Lighthill's analogy, yields the integral formulation of Curle (1955). The control volume is chosen such that it encloses the observation point $\vec{x}$. Note that in the literature the sign of the unit normal $\vec{n}$ is often chosen to be the opposite of the sign chosen here [Goldstein (1976), Dowling and Ffowcs Williams (1983)].

An alternative approach is the use of a so-called tailored Green function [Dowling and Ffowcs Williams (1983)]. This is a Green function defined by the wave equation (71) and the same (locally reacting linear) boundary conditions as the acoustic field under consideration. In that case the surface integrals of (88) vanish. An example of such a Green function for the trailing edge of a plate will be discussed in later chapters, Part 2.

4.5 Monopole, dipole and quadrupole

We consider radiation of a spatially limited source-region under free field conditions. Whenever the source region ($q(\vec{x},t) \neq 0$) is compact, we can neglect variations in the retarded time t_e in the integral of equation (87).

Choosing the origin within the source region we get at distances large compared to the source region:

$$r = |\vec{x} - \vec{y}| \approx |\vec{x}| \tag{89}$$

and

$$t_e \approx t - \frac{|\vec{x}|}{c_0} \tag{90}$$

so that we have:

$$p'(\vec{x}, t) \approx \frac{1}{4\pi|\vec{x}|} \int_V q\left(\vec{y}, t - \frac{|\vec{x}|}{c_0}\right) dV_y \ . \tag{91}$$

We call the integral $\int_V q\left(\vec{y}, t - \frac{|\vec{x}|}{c_0}\right) dV_y$ the monopole strength of the source region. Whenever the source is the divergence of a force field $q(\vec{x}, t) = -\nabla\cdot\vec{f}$ integral (91) taken over a volume including the source region will vanish because the surface integral of the flux of the force field $\int_S \vec{f}\cdot\vec{n}dS_y = 0$ vanishes because $\vec{f} = 0$ on the surface. The surface, including the control volume, is outside the source region so that the force is either uniform or zero. By partial integration and using the symmetry property (83) we can write the formal solution of the wave equation as:

$$\begin{aligned} p'(\vec{x}, t) &= -\int_{-\infty}^{t}\int_V (\nabla_y\cdot\vec{f}(\vec{y}, \tau))G_0(\vec{x}, t|\vec{y}, \tau)dV_y d\tau = \\ &= -\int_{-\infty}^{t}\int_V \vec{f}(\vec{y}, \tau)\nabla_x G_0(\vec{x}, t|\vec{y}, \tau)dV_y d\tau \ . \end{aligned} \tag{92}$$

As the integration over the source coordinates $\vec{y}$ does not interfere with the derivation by observer's coordinates $\vec{x}$ we have:

$$p'(\vec{x}, t) = -\nabla_x\cdot\int_{-\infty}^{t}\int_V \vec{f}(\vec{y}, \tau)G_0(\vec{x}, t|\vec{y}, \tau)dV_y d\tau \ . \tag{93}$$

For a compact source $(k|\vec{y}| << 1$ and distances large compared to the dimension of the source region $(|\vec{x}| >> |\vec{y}|)$, we have a dipole field:

$$p'(\vec{x}, t) \approx -\nabla_x\cdot\left(\frac{1}{4\pi|\vec{x}|}\int_V \vec{f}\left(\vec{y}, t - \frac{|\vec{x}|}{c_0}\right) dV_y\right) \tag{94}$$

where $\left(\int_V \vec{f}\left(\vec{y}, t - \frac{|\vec{x}|}{c_0}\right) dV_y\right)$ is the dipole strength.
An alternative way to find this expression is to consider the solution ϕ_i of the wave equation:

$$\frac{1}{c_o^2}\frac{\partial^2\phi_i}{\partial t^2} - \nabla^2\phi_i = -f_i \tag{95}$$

following (87) this is simply:

$$\phi_i(\vec{x},t) = -\int_{-\infty}^{t}\int_V f_i(\vec{y},\tau)G_0(\vec{x},t|\vec{y},\tau)dV_y d\tau \ . \tag{96}$$

Obviously taking the divergence of equation 95 we also have:

$$\frac{1}{c_0^2}\frac{\partial^2(\partial\phi_i/\partial x_i)}{\partial t^2} - \nabla^2(\partial\phi_i/\partial x_i) = -\frac{\partial f_i}{\partial x_i} \tag{97}$$

which leads to equation (93) because $p'(\vec{x},t) = -\nabla\cdot\vec{\phi}$.

While a monopole can be represented as a pulsating compact sphere, a dipole field is generated by a compact translating sphere. In a similar way we can obtain for the sound source $\frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j}$ found in the analogy of Lighthill:

$$p'(\vec{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_{-\infty}^{t}\int_V \rho v_i v_j G_0(\vec{x},t|\vec{y},t)dV_y d\tau \ . \tag{98}$$

In a compact source region this is a so-called quadrupole field.

An alternative approach to the multipole expansion of the source [Goldstein (1976)] is to use a Taylor series expansion of the free space Green function around $\vec{y}=0$ in the general solution (87):

$$\begin{aligned} p'(\vec{x},t) = \textstyle\int_{-\infty}^{t}\int_V q(\vec{y},\tau)\Big[G_0(\vec{x},t\ 0,\tau) + \left(\frac{\partial G_0}{\partial y_i}\right)_{\vec{y}=0} y_i + \\ + \tfrac{1}{2}\left(\frac{\partial^2 G_0}{\partial y_i \partial y_j}\right)_{\vec{y}=0} y_i y_j + ...\Big] dV_y d\tau \ . \end{aligned} \tag{99}$$

which, using the symmetry properties (85) of the Green's function and the far field approximation, yields:

$$\begin{aligned} p'(\vec{x},t) \approx \tfrac{1}{4\pi|\vec{x}|}\textstyle\int_V q\left(\vec{y},t-\frac{|\vec{x}|}{c_0}\right)dV_y + \frac{x_i\partial}{4\pi|\vec{x}|^2 c_0 \partial t}\int_V y_i q(\vec{y},t-\frac{|\vec{x}|}{c_0})dV_y + \\ + \tfrac{x_i x_j}{4\pi|\vec{x}|^3}\frac{\partial^2}{c_0^2\partial t^2}\textstyle\int_V \frac{1}{2}y_i y_j q\left(\vec{y},t-\frac{|\vec{x}|}{c_0}\right)dV_y + ... \end{aligned} \tag{100}$$

An intuitive interpretation of monopole, diopole and quadrupole on surface water waves is provided in Figure 4. Due to the oscillating momentum in the region between the two monopoles forming a dipole it is obvious that a dipole cannot exist without any force acting on the fluid. This force is needed to change the momentum. Thus, unsteady force induces a dipole radiation and a dipole radiation cannot exist without a force acting on the fluid.

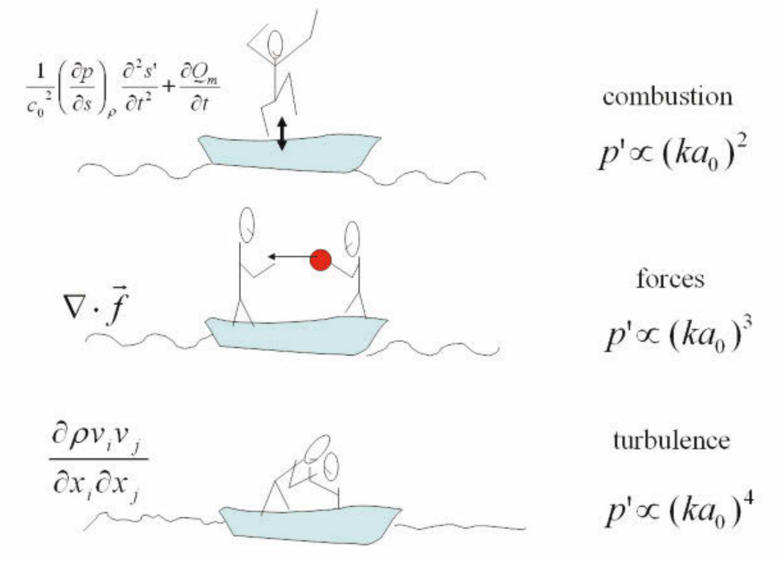

Figure 4. Intuitive interpretation of monopole, diopole and quadrupole on surface water waves. We provide a qualitative interpretation of physical realisations of monopole, dipole and quadrupole. Mass exchange between two monopoles with opposite phase implies an oscillating momentum. This is impossible without external force.

4.6 Analogy of Curle

The analogy of Curle (1955) is the integral formulation (88) applied to Lighthill's analogy (42) in terms of density fluctuations:

$$\begin{aligned} p'(\vec{x},t) = c_0^2\rho'(\vec{x},t) = \int_{-\infty}^{t}\int_V \left(\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} - \frac{\partial f_i}{\partial y_i}\right) G_0(\vec{x},t|\vec{y},\tau)dV_y d\tau \\ -c_0^2\int_{-\infty}^{t}\int_S \left[\rho'\frac{\partial G_0}{\partial y_i} - G_0\frac{\partial \rho'}{\partial y_i}\right] n_i dS_y d\tau \ . \end{aligned} \tag{101}$$

The observer is placed within the control volume V over which we carry out the integration. This equation is based on the assumption that at the listener's position $p' = c_0^2\rho'$. We will further ignore the contribution from the external force field ($\vec{f} = 0$). By means of partial integration we move the space derivatives from the source terms towards the Green function:

$$\begin{aligned} p'(\vec{x},t) = c_0^2 p' = \int_{-\infty}^{t} \int_V T_{ij} \frac{\partial^2 G_0}{\partial y_i \partial y_j} dV_y d\tau + \\ + \int_{-\infty}^{t} \int_S \left[G_0 \frac{\partial T_{ij}}{\partial y_i} n_j - T_{ij} \frac{\partial G_0}{\partial y_j} n_i \right] dS_y d\tau - \\ - c_0^2 \int_{-\infty}^{t} \int_S \left[\rho' \frac{\partial G_0}{\partial y_i} - G_0 \frac{\partial \rho'}{\partial y_i} \right] n_i dS_y d\tau \ . \end{aligned} \tag{102}$$

Using the definition of the viscous stress tensor (26) and the momentum equation (9) we can write (102) in the form:

$$\begin{aligned} p'(\vec{x},t) = c_0^2 p' = \int_{-\infty}^{t} \int_V T_{ij} \frac{\partial^2 G_0}{\partial y_i \partial y_j} dV_y d\tau + \int_{-\infty}^{\infty} \int_S G_0 \frac{\partial \rho v_i}{\partial t} n_i dS_y d\tau \\ - \int_{-\infty}^{t} \int_S \left(P_{ij} + \rho v_i v_j \right) \frac{\partial G_0}{\partial y_j} n_i dS_y d\tau + \int_{-\infty}^{t} \int_S \left(p' - c_0^2 \rho' \right) \frac{\partial G_0}{\partial y_i} n_i dS_y d\tau \ . \end{aligned} \tag{103}$$

Furthermore we neglect entropy fluctuations on the surface S.

By means of partial integration we move the time derivative in the second integral from the momentum flux to the Green's function. Using the symmetry relations of the free field derivative with respect to space (85) and time derivatives (84), we find in the far field approximation (60):

$$\begin{aligned} p'(\vec{x},t) = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int_S \left[\frac{\rho v_i}{r} \right]_{\tau = t_e} n_i dS_y - \\ - \frac{x_j}{4\pi |\vec{x}| c_0} \frac{\partial}{\partial t} \int_S \left[\frac{P_{ij} + \rho v_i v_j}{r} \right]_{\tau = t_e} n_i dS_y + \\ + \frac{x_i x_j}{4\pi |\vec{x}|^2 c_0^2} \frac{\partial^2}{\partial t^2} \int_V \left[\frac{T_{ij}}{r} \right]_{\tau = t} dV_y \ . \end{aligned} \tag{104}$$

In (104) we recognize the monopole sound production due to the volume flux leaving the surface (first integral), the dipole field generated by the force acting on the surfaces and the quadrupole field generated by fluctuations of the Reynolds stress tensor in the volume.

4.7 Doppler Effect

In Curle's formulation (section 4.6) we restricted ourselves to fixed control volumes. When considering sound produced by moving objects such as fan blades, it is more convenient to use a moving control volume. Ffowcs Williams and Hawkings (1969b) use generalized functions to take into account the motion of the sources, the result being a generalization of Curle's equation in which Doppler factors appear. In a further step Ffowcs Williams and Hawking [Goldstein (1976), Dowling and Ffowcs Williams (1983), Crighton et al. (1992)] introduce the boundaries of the control volume in the equation of motion, see next chapter. We now focus on the derivation

of the Doppler effect for point sources.
A moving point source is described by:

$$q(\vec{x},t) = Q(t)\delta(\vec{x}-\vec{x}_s(t)) \tag{105}$$

where $\vec{x}_s(t)$ is the position of the source. For free-field conditions we have:

$$p'(\vec{x},t) = \int_{-\infty}^{t}\int_V \frac{Q(\tau)\delta(\vec{y}-\vec{x}_s(\tau))\delta(\tau-t_e)}{4\pi r} dV_y d\tau \tag{106}$$

where $t_e = t - \frac{r}{c_0}$ and $r = |\vec{x}-\vec{y}|$. Using the properties of the delta function we get after spatial integration:

$$p'(\vec{x},t) = \int_{-\infty}^{t} \frac{Q(\tau)\delta\left(t-\tau-\frac{|\vec{x}-\vec{x}_s(\tau)|}{c_0}\right)}{4\pi|\vec{x}-\vec{x}_s(\tau)|} d\tau \tag{107}$$

This is an integral of the type:

$$\int_{-\infty}^{\infty} F(\tau)\delta(H(\tau))d\tau = \Sigma_i \frac{F(t_i)}{|\frac{dH}{d\tau}|_{\tau=t_i}} \tag{108}$$

with $H(t_i) = 0$. In the present case we have:

$$H(\tau) = t - \tau \frac{|\vec{x}-\vec{x}_s(\tau)|}{c_0} \tag{109}$$

so that:

$$\frac{dH}{d\tau} = -1 + \frac{\vec{x}-\vec{x}_s}{c_0|\vec{x}-\vec{x}_s(\tau)|}\cdot\frac{d\vec{x}_s}{d\tau} = -1 + M_r \tag{110}$$

where M_r is the ratio of the source velocity component in the direction of the observer and the speed of sound. The sound field is given by:

$$p'(\vec{x},t) = \frac{Q(t_e)}{4\pi|1-M_r||\vec{x}-\vec{x}_s(t_e)|} \tag{111}$$

where the emission time is the root of:

$$c_0(t-t_e) = |\vec{x}-\vec{x}_s(t_e)|\ . \tag{112}$$

For subsonic velocities there is only one root ($\tau = t_e$) of $H(\tau) = 0$. For a harmonically oscillating sound source with constant frequency ω, the frequency of the signal reaching the observer is:

$$\frac{d\omega t_e}{dt} = \frac{\omega}{1-M_r(t_e)} \tag{113}$$

because $\frac{dt_e}{dt} = \frac{1}{1-M_r(t_e)}$.

A further discussion of the Doppler Effect is provided in the next chapter, where it is shown that for supersonic Mach numbers, the sound source will have a strong radiation for directions such that $M_r = 1$. An example of such a radiation occurs when elastic bending waves in a plate propagate supersonically with respect to the surrounding fluid. As the velocity of propagation of bending waves increases with the frequency this occurs typically above a critical frequency f_c, which is called the coincidence frequency. This explains why we hear a high pitch when we hit a glass window.

From equation (111) we observe that in addition to the change in frequency we have an effect of the source motion on the amplitude reaching the observer. This effect can be understood as a result of the change in ratio of source size to acoustic wave length. From equation (63) we know that with increasing Helmholtz numbers the radiate sound amplitude of a compact object increases. In the direction of motion of the source, the emitted acoustic wave length is shorter by a factor $1 - M_r$, with an increased effective Helmholtz number as a result. In figure 5 we provide an intuitive interpretation of the Doppler shift in frequency.

Furthermore we note that for a moving object of volume V the sound source is $q(\vec{x}, t) = \rho_0 \frac{d^2V}{dt^2} \delta(\vec{x} - \vec{x}_s(t))$. Hence we have:

$$p'(\vec{x}, t) = \rho_0 \frac{\partial^2}{\partial t^2} \left(\frac{V(t_e)}{4\pi|1 - M_r||\vec{x} - \vec{x}_s(t_e)|} \right) . \tag{114}$$

It shows that due to the time dependency of the retarded time $\partial t_e/\partial t$ an object of constant volume will radiate sound if its velocity varies. This is the so called thickness noise p'_{th}, which is very important in aircraft fans. In the far field approximation for a rigid of volume V body moving at subsonic speed, we have:

$$p'_{th}(\vec{x}, t) \approx \rho_0 V \left(\frac{|1 - M_r| \frac{d^2 M_r}{dt_e^2} + 3 \frac{dM_r}{dt_e}}{4\pi|1 - M_r|^3 (1 - M_r)^2} \right) \frac{1}{|\vec{x} - \vec{x}_s(t_e)|} . \tag{115}$$

Another example is the sound radiated by a moving point force:

$$\vec{f} = \vec{F}(t)\delta(\vec{x} - \vec{x}_s(t)) \tag{116}$$

which is given by:

$$p'(\vec{x}, t) = -\nabla \cdot \left(\frac{\vec{F}(t_e)}{4\pi|\vec{x} - \vec{x}_s(t_e)||1 - M_r|} \right) . \tag{117}$$

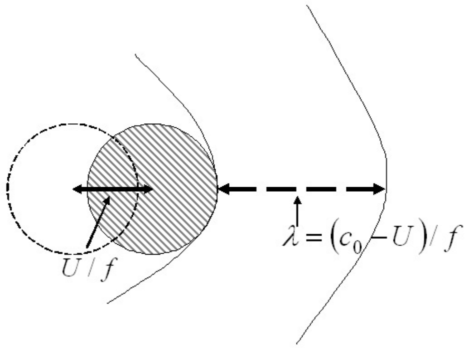

Figure 5. Intuitive interpretation of Doppler effect as a change in wave length $\lambda = \frac{c_0 - U}{f}$ of radiated wave due to the movement of the source with a velocity U in the direction of the listener. The wave-length is reduced in the direction of the movement. This implies a reduction of the compactness of the source and leads to an increased radiation power.

In the far-field approximation we have:

$$p'(\vec{x},t) = -\frac{1}{1-M_r}\frac{(\vec{x}-\vec{x}_s(t_e))}{|(\vec{x}-\vec{x}_s(t_e))|}\cdot\frac{\partial}{\partial t_e}\left(\frac{\vec{F}(t_e)}{4\pi|\vec{x}-\vec{x}_s(t_e)||1-M_r(t_e)|}\right). \tag{118}$$

4.8 Influence of speed of sound gradient and of convective effects

Whenever a source of sound is compact we can separate the sound generation from the wave propagation. Even with this simplification the wave propagation remains extremely complex.

In the presence of flow and gradients in the speed of sound, acoustic waves display complex propagation behaviour [Dowling 1983, Pierce 1990, Rienstra 1999]. An example of this is the sound propagation in the atmosphere. As a result of the non-uniformity of the temperature in the atmosphere waves are deflected from the straight path assumed in the elementary solutions for uniform stagnant fluid. An example being that a gun shot or thunder heard at large distances can be repeated multiple times, which yields a roll sound. This is due to the fact that sound can reach our ears along multiple paths.

We now consider a very basic problem of a plane wave that is reflected at the interface (shear layer) between two uniform media a and b with each having uniform flow speeds respectively $\vec{U}_a = (U_a, 0, 0)$ and $\vec{U}_b = (U_b, 0, 0)$ and speeds of sound c_a and c_b respectively.

In presence of a uniform flow the plane wave solution (52) becomes:

$$p'(\vec{x}, t) = A \exp\left(i\omega\left(t - \frac{\vec{x}\cdot\vec{n}}{c_0 + \vec{n}\cdot\vec{U}}\right)\right) = A\exp\left(i\left(\omega t - \vec{k}\cdot\vec{x}\right)\right) \tag{119}$$

with $\vec{n} = (\cos\theta, \sin\theta, 0)$ and $\vec{k} = \omega\vec{n}/(c_0 + \vec{n}\cdot\vec{U})$.

We assume an incident wave with amplitude I and wave number $\vec{k}_I = \frac{\omega\vec{n}}{c_a + \vec{n}_I \cdot \vec{U}_a}$ in region a. This induces a reflected wave with amplitude R and wave number $\vec{k}_R = \frac{\omega\vec{n}_R}{c_a + \vec{n}_R\cdot\vec{U}_a}$ and a transmitted wave with amplitude T and wave number $\vec{k}_T = \frac{\omega\vec{n}_T}{c_b + \vec{n}_T\cdot\vec{U}_b}$ (Figure 6).

At the interface $x_2 = 0$ we have continuity of pressure so that for $x_2 = 0$ we have:

$$\begin{aligned} &I\exp\left(-\omega\frac{x_1\cos\theta_I}{c_a + U_a\cos\theta_I}\right) + R\exp\left(-i\omega\frac{x_1\cos\theta_R}{c_a + U_a\cos\theta_R}\right) = \\ &= T\exp\left(-i\omega\frac{x_1\cos\theta_T}{c_b + U_b\cos\theta_T}\right)\ . \end{aligned} \tag{120}$$

As this equation should hold for any value of the coordinate x_1 (along the shear layer) the exponents should be identical:

$$\left(\frac{\cos\theta_I}{c_a + U_a\cos\theta_I}\right) = \left(\frac{\cos\theta_R}{c_a + U_a\cos\theta_R}\right) = \left(\frac{\cos\theta_T}{c_b + U_b\cos\theta_T}\right)\ . \tag{121}$$

The first equality of (121) implies that $\cos\theta_1 = \cos\theta_R$, so that the reflection angle is equal to the incidence angle $\theta_R = -\theta_I$.
The second equality of (121) yields the modified Snelius law:

$$\frac{c_a}{\cos\theta_I} + U_a = \frac{c_b}{\cos\theta_T} + U_b \tag{122}$$

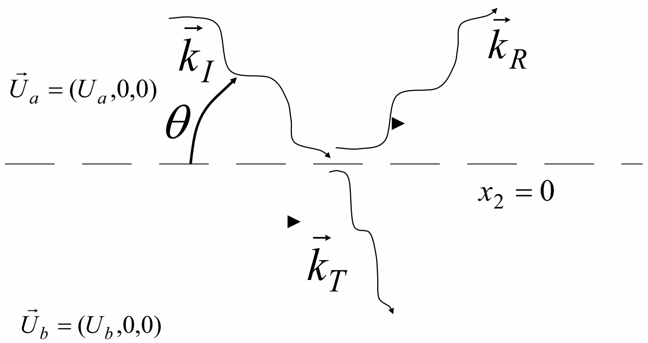

Figure 6. Reflection and refraction of a plane wave at a flat shear layer $x_2 = 0$ separating two uniform flows.

or:

$$\cos\theta_T = \frac{c_b \cos\theta_I}{c_a + (U_a - U_b)\cos\theta_I} \ . \tag{123}$$

The maximum transmission angle is found for grazing incidence $\cos\theta_I = 1$: In the particular case of $c_a = c_b$ and $U_b = 0$ we find:

$$(\theta_T)_{max} = arcos\left(\frac{1}{1 + (U_a/c_a)}\right) \tag{124}$$

In high speed jets one does indeed observe a cone of silence along the axis of the jet, because the acoustic waves emitted along the main flow direction are bent away from the flow direction by the velocity gradient in the shear layers [Morfey (1978)].

The amplitude of the transmitted and reflected waves is calculated from the continuity of pressure at the interface $I + R = T$ complemented by the continuity of particle displacement at the interface.

5 Turbulence noise at low Mach numbers

5.1 Isothermal free jet

Considering the sound production of a turbulent free jet. This is the flow with a velocity U_0 at the outlet of a pipe of diameter D. Turbulence is an unsteady chaotic fluid motion which appears when viscous forces are small compared to non-linear convective forces. This corresponds to high Reynolds numbers $Re_D = U_0 D/\nu$. We limit ourselves to a low Mach number flow $M = U_0/c_0 \ll 1$ of an air jet surrounded by air with the same temperature as its surroundings. The prediction of the scaling rule between the power of this sound source and the Mach number was a major success of the theory of Lighthill (1952-54). As stressed by Powell (1990), the scaling law was predicted before it was corroborated by experiments. The steps taken by Lighthill were, however, quite intuitive and justification of some of these steps came only long after the original publication [Morfey (1973),(1976),(1978), Obermeier (1975)]. We now follow the Lighthill prodecure [1954].

Firstly Lighthill assumes that there are no external forces working on the flow and that the effect of walls can be neglected. In free field conditions equation (99) simplifies to:

$$\begin{aligned} p'(\vec{x},t) &= c_0^2\rho' \int_{-\infty}^{t}\int_V T_{ij}\frac{\partial^2 G_0}{\partial y_i \partial y_j} dV_y d\tau = \\ &= \frac{\partial^2}{\partial x_i \partial x_j}\int_{-\infty}^{t}\int T_{ij}\frac{\delta\left(t-\tau-\frac{r}{c_0}\right)}{4\pi r} dV_y d\tau \ . \end{aligned} \tag{125}$$

This implies that the solution we are seeking for is, at most, a quadrupole field. In fact, we have imposed this by assuming that there are no external forces acting on the fluid and the potential monopole sources were neglected. Please note that in the analogy of Lighthill, ρ' is used as aeroacoustical variable. In the next section we will discuss why this choice can be important. Carrying the time integration and using the far field approximation we find:

$$p'(\vec{x},t) = c_0^2\rho' = \frac{x_i x_j}{|\vec{x}|^2 c_0^2}\frac{\partial^2}{\partial t^2}\int_V \frac{T_{ij}(\vec{y}, t-\frac{r}{c_0})}{4\pi r} dV_y \ . \tag{126}$$

The sound appears to be produced mainly by large coherent vortex structures with a length scale of the order of the pipe diameter D. For such scales the Reynolds number is large. We therefore expect the Reynolds stress tensor $\rho v_i v_j$ to be much larger than the viscous stress tensor τ_{ij} [Morfey (1976)]. Furthermore, at low Mach numbers variations in temperature and density are negligible [Morfey (1973), Morfey et al. (1978)], which implies that we

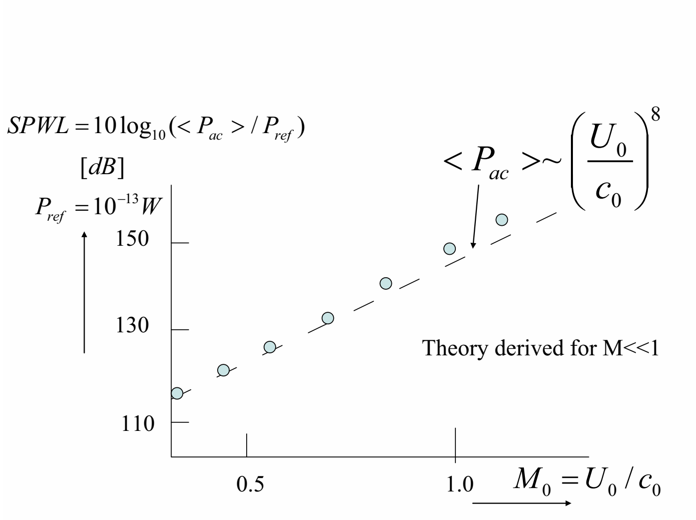

Figure 7. Overall acoustic sound power level (OAPWL) of the sound radiation from an isothermal free jet as a function of the jet Mach number: comparison of theory with experimental results [Fisher et al. (1973),Viswanathan (2009)].

can use the approximation proposed by Lighthill (1952-54):

$$T_{ij} \approx \rho_0 v_i v_j \ . \tag{127}$$

For a circular jet cross section the dominant frequency corresponds to a Strouhal number of unity. Hence the dominating frequency is U_0/D and the corresponding acoustic wavelength is $D/M = Dc_0/U_0$. The sound source has a volume V of the order of D^3. At low Mach numbers the sound source is small compared to the wave length. This implies that we can neglect variations of the retarded time in the integral (126): $r = |\vec{x - y}| \approx |\vec{x}|$. Summarizing we use the scaling rules:

$$\frac{\partial}{\partial t} \sim \frac{U_0}{D} \tag{128}$$

$$T_{ij} \sim \rho_0 U_0^2 \tag{129}$$

$$V \sim D^3 \ . \tag{130}$$

Substitution in (126) yields:

$$p' \sim \frac{\rho_0 U_0^4}{c_0^2}\left(\frac{D}{r}\right) . \tag{131}$$

In terms of sound source power $< P >= 4\pi r^2 \frac{(p')^2}{\rho_0 c_0}$ we have:

$$\frac{< P >}{\frac{1}{2}\rho_0 U_0^3 \frac{\pi D^2}{4}} \sim 32M^5 \tag{132}$$

where we assumed an isotropic radiation pattern. This famous global scaling rule of Lighthill (1952-54) appears to be valid up to Mach numbers of order unity. At these high Mach numbers the radiation pattern has a high forward directivity due to the Doppler effect and, due to refraction of sound by the shear layers, it displays a cone of silence around the axis. The fact that the theory remains valid up to relatively high Mach numbers can be partially explained by the fact that the convection velocity U_c of the vortices in the jet is only a fraction of the main flow velocity [Crighton et al. (1992)]. Typically we have $U_c/U_0 \approx 0.3$. Recent discussions on jet noise are Morris and Farassat (2002) and Viswanathan (2009) as well as the discussion in Part 2.

Obviously, by increasing the Mach number, the scaling law of Lighthill fails simply because the radiated power would become larger than the available jet power $\frac{1}{2}\rho U_0^3 \frac{\pi D^2}{4}$. Also the sound production mechanism changes drastically. The sound radiation from supersonic jets above $M = 3$ is largely due to hydrodynamic shear waves which display highly directional radiation patterns. Entropy effects due to temperature differences in the flow also become very important. In a supersonic flow the temperature typically varies from the stagnation temperature T_t to the isentropic expansion temperature $T = T_t/(1 + (\gamma - 1)M^2/2)$. Starting from room temperature $T_t \approx 300K$ in the reservoir, $M = 3$ implies a main flow temperature $T \approx 100K$. Obviously, such a flow is not isothermal and we can use many different definitions of the temperature or Mach numbers to characterize the flow [Viswanathan (2009)].

Finally, most supersonic jet are either over- or underexpanded, and therefore display standing shock structures, which interact with vortices (turbulence) that give strong sound radiation. In some cases, this leads to spectacular self-sustained oscillation (jet screetch).

Note that approximation (128) is based on the fact that in a circular jet the characteristic Strouhal number for the sound production is of order unity $Sr_D = Df/U = 0(1)$. In a planar jet of thickness H we find

$Sr_H = Hf/U = 0(10^{-1})$, which again stresses that the assumptions are not trivial [Bjørnø et al. (1984)].

Turbulence noise is essential because, when all other sound sources have been suppressed, this will always remain as the minimum remaining noise production which we can achieve. Lighthill's scaling law indicates that the most efficient way to reduce this noise is to reduce the flow velocity. The result derived for free-field conditions remains valid for confined flow. In the absence of resonances, one finds at low frequencies in a pipe $p' \sim \rho_0 U_0^3/c_0$ and $< P > \sim M^6$.

It is important to stress again that the analogy of Lighthill does not impose the quadrupole character of the source. Because we neglected the monopoles (no heat sources and negligible variation in density) and the dipoles (no external force acting on the "free" jet), the source has at most, a quadrupole character. Based on the integral formulation (126) the procedure imposes this assumed quadrupole character on the solution. So even if the applied model predicting the stress tensor T_{ij} does involve density fluctuations and external forces, the formulation ensures that these contributions are ignored. This explains the success of such analogies [Schram and Hirschberg (2003)]. They filter out spurious sound sources due to errors in the estimation of the stress tensor T_{ij}.

5.2 Bubbly liquids

In the previous sections we used the analogy of Lighthill (1952-54) to obtain a scaling law for sound production by subsonic isothermal free jets. One of the choices in this derivation is to express the analogy in terms of fluctuations of density ρ' (equation 40). As an alternative, we could have also used the fluctuations of pressure p' (equation 39). In principle both formulations are equivalent as long as no approximations are involved. However, an analogy is only meaningful if we do use approximations. Depending on the choice of the aero-acoustic variable some approximations will appear naturally. For example using the pressure formulation, the entropy noise source term has the form $\partial^2(p'/c_0^2 - \rho')/\partial t^2$. This is a monopole sound source, to be understood as the time dependent volume expansion due to unsteady combustion. A more detailed analysis of thermal effects is provided by Morfey et al. (1978) and Dowling [in, Crighton et al. (1992)]). Using the density formulation, the entropy sound source term is a spatial derivative $\partial^2(p' - c_0^2\rho')/\partial x_i^2$. We will now explain the physical meaning of this apparently obscure sound source term. For this we consider the sound produced by a turbulent free jet in a bubbly liquid, as observed by a listener immerged in the pure liquid. In such a case the speed of sound c in the source region is much lower than the speed of sound c_0 of the fluid surrounding the listener.

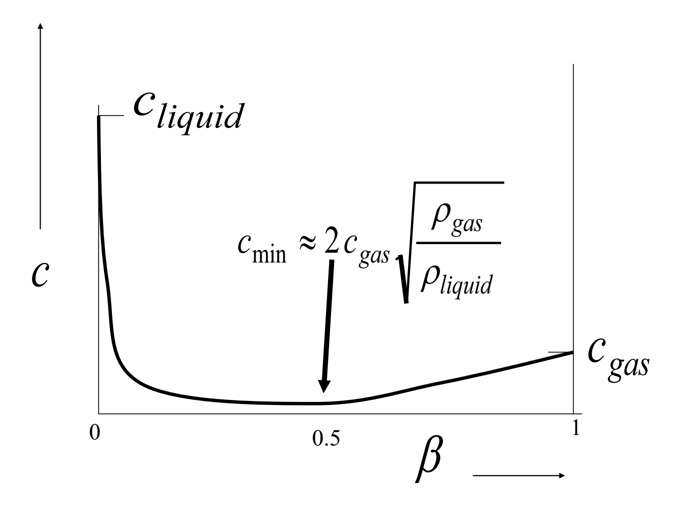

Figure 8. Low frequency limit of the speed of sound in a bubbly liquid as a function of the gas volume fraction [Crighton et al. (1992)].

Considering the low frequency limit of the behaviour of a mixture of gas bubbles and a liquid (Figure 8). We find that low frequency implies that gas density ρ_g and fluid density ρ_l are both uniform so that the mixture density ρ is given by [Crighton et al. (1992)]:

$$\rho = \beta\rho_g + (1 - \beta)\rho_l \tag{133}$$

where β is the volume fraction of gas in the mixture. Assuming a quasi-steady behaviour, the pressure is uniform. Thus, we can add the compressibility of the two phases to obtain the compressibility of the mixture:

$$\frac{1}{\rho c^2} = \frac{\beta}{\rho_g c_g^2} + \frac{(1 - \beta)}{\rho_l c_l^2} \tag{134}$$

where c_g is the speed of sound in gas and c_l is the speed of sound in liquid.

Eliminating the density by multiplying (133) by (134) yields:

$$c^2 = \frac{1}{[\beta\rho_g + (1-\beta)\rho_l]\left[\frac{\beta}{\rho_g c_g^2} + \frac{(1-\beta)}{\rho_l c_l^2}\right]} \ . \tag{135}$$

For air/water mixtures at neither too small or too large a value of β we can neglect both the contribution of air to the mass density and the contribution of water to the compressibility. We then get:

$$c^2 \approx \frac{1}{[(1-\beta)\rho_l]\left[\frac{\beta}{\rho_g c_g^2}\right]} = c_g^2 \left(\frac{\rho_g}{\rho_l}\right) \frac{1}{\beta(1-\beta)} \ . \tag{136}$$

For air with ($\rho_g = 1.2 kg/m, c_g = 340 m/s$) and water with ($\rho_l = 1000 kg/m$, $c_l = 1500 m/s$) we get a minimal speed of sound $c_{min} \approx 20 m/s$ at $\beta = 0.5$. The entropy term in the analogy of Lighthill for an isentropic flow can be written as follows:

$$\frac{\partial^2}{\partial x_i^2}(p' - c_0^2\rho') = \frac{\partial^2}{\partial x_i^2} p' \left(1 - \frac{c_0^2}{c^2}\right) \ . \tag{137}$$

The pressure fluctuations in the source region are of the same order as the fluctuations in the Reynolds stress tensor: $p' \sim \rho U^2$. Hence, compared to a free jet of water surrounded by water, the bubbly liquid turbulence sound is enhanced by a factor $|(1 - c_0^2/c^2)| = 5 \times 10^3$, which is 74 dB. Infact, taking a shower in a bath tub, we observe that the water jet impinging on the water surface is much noisier than the jet immerged in the water, as we can understand qualitatively in terms of the analogy of Lighthill. According to Morfey (1973) and Powell (1990) this entropy term can be understood as the sound produced by the unsteady force exerted on the mixture as a result of the "buoyancy" force due to the difference in density between the two phases undergoing a pressure gradient. This corresponds to a slip between the two phases. Obviously, as there are no net external forces, this sound source must be a quadrupole, the force of the gas on the liquid being balanced by the reaction force of the liquid on the gas.

Similar effects, though much weaker can be found in non-isothermal gas free jets. Contrary to earlier literature predicting a dipole [Morfey (1973), Obermeier (1975)], recent studies indicate that the overall acoustic power level radiated by hot jets is also in line with the height power law of Lighthill [Viswanathan (2009)], which actually confirms that this sound source is also a quadrupole. In the early literature it was also suggested that next to convection effects due to density differences, the heat transfer between a hot gas free jet and its surroundings would generate a monpole sound source. In cases with ideal gasses and a uniform constant Poisson ratio γ, this does not occur due to the jet contraction by cooling compensating exactly expansion of the surroundings due to heating [Morfey and Wright (2007)]. Monopole sound sources do occur as a result of combustion or phase transition (moisture condensation).

Bubble resonance can induce an even larger amplification of turbulent sound production [Dowling and Ffowcs Williams (1983)]. Yet, it is argued by Crighton (1975) that typical turbulent eddies corresponding to frequencies close to resonance frequencies of bubbles are much smaller than the bubbles and can therefore not excite the bubbles coherently. He therefore uses the low frequency approximation described above.

6 Waves in pipes

6.1 Pipes modes

We are considering propagation of harmonic waves $p' = \hat{p}\exp(i\omega t)$ in a duct with a uniform rectangular cross section, with the duct axis is in the x_3 direction. The duct is delimited by rigid walls in the planes: $x_1 = 0, x_1 = h_1, x_2 = 0, x_2 = h_2$ (Figure 9). For such harmonic waves the wave equation

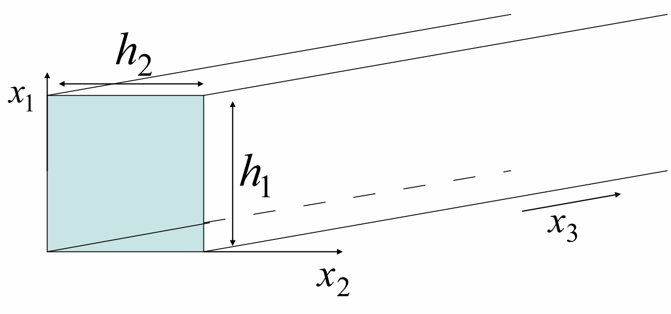

Figure 9. Duct with rectangular cross section.

(47) can be written as:

$$\left[k_0^2 + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right]\hat{p} = 0 \ . \tag{138}$$

This is the Helmholtz equation.
Seeking a solution by using the method of separation of variables:

$$\hat{p} = F(x_1)G(x_2)H(x_3) \ . \tag{139}$$

and substituting (139) in (138) we get:

$$k_0^2 + \frac{1}{F}\frac{d^2F}{dx_1^2} + \frac{1}{G}\frac{d^2G}{dx_2^2} + \frac{1}{H}\frac{d^2H}{dx_3^2} = 0 \ . \tag{140}$$

As this equation should be valid for any value of $\vec{x} = (x_1, x_2, x_3)$ each factor in (140) should be constant:

$$\frac{1}{F}\frac{d^2F}{dx_1^2} = -\alpha^2 \ . \tag{141}$$

$$\frac{1}{G}\frac{d^2G}{dx_2^2} = -\beta^2 \tag{142}$$

and

$$\frac{1}{H}\frac{d^2H}{dx_1^2} = -\left[k_0^2 - \alpha^2 - \beta^2\right] \ . \tag{143}$$

The constants α and β are determined by the boundary conditions of zero normal velocity at the rigid walls. The normal component of the pressure gradient, which is proportional to this normal velocity, vanishes at the walls:

$$\left(\frac{dF}{dx_1}\right)_{x_1=0} = \left(\frac{dF}{dx_1}\right)_{x_1=h_1} = 0 \tag{144}$$

and

$$\left(\frac{dG}{dx_2}\right)_{x_2=0} = \left(\frac{dG}{dx_2}\right)_{x_2=h_2} = 0 \tag{145}$$

From this we can conclude that the possible solutions for F and G have the form:

$$F_m = \cos(\alpha_m x_1) \ \ ; \ \ \alpha_m = \frac{m\pi}{h_1} \ \ ; \ \ m = 0, 1, 2, 3, ... \tag{146}$$

and

$$G_n = \cos(\beta_n x_2) \ \ ; \ \ \beta_n = \frac{n\pi}{h_2} \ \ ; \ \ n = 0, 1, 2, 3, ... \tag{147}$$

Substitution in equation (143) yields:

$$\frac{1}{H_{mn}}\frac{d^2H_{mn}}{dx_1^2} = -\left[k_0^2 - \alpha_m^2 - \beta_n^2\right] = -k_{mn}^2 \; . \tag{148}$$

There are two types of solution, depending on the sign of k_{mn}^2. For positive values we have propagating wave modes:

$$\hat{p}_{mn}^{\pm} = \cos\left(\frac{m\pi}{h_1}x_1\right)\cos\left(\frac{n\pi}{h_2}x_2\right)\exp(\mp i|k_{mn}|x_3) \tag{149}$$

and for negative values we have evanescent modes:

$$\hat{p}_{mn}^{\pm} = \cos\left(\frac{m\pi}{h_1}x_1\right)\cos\left(\frac{n\pi}{h_2}x_2\right)\exp(\mp|k_{mn}|x_3) \tag{150}$$

with

$$|k_{mn}| = \left|\sqrt{k_0^2 - \left(\frac{m\pi}{h_1}\right)^2 - \left(\frac{n\pi}{h_2}\right)^2}\right| \tag{151}$$

The solution we are looking for is a linear superposition of these modes:

$$p' = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(A_{mn}^{+}\hat{p}_{mn}^{+} + A_{mn}^{-}\hat{p}_{mn}^{-}\right)\right)\exp(i\omega t) \tag{152}$$

where the amplitudes of the modes are determined by the boundary conditions at the boundaries of the duct in the x_3 direction. For each mode there is a cut off frequency $(\omega_{mn})_c$ below which the mode is evanescent. For example for the mode $\hat{p}_{10}$ we have:

$$(\omega_{10})_c = \frac{\pi c_0}{h_1} \; . \tag{153}$$

The duct width should be larger than half the wave length to allow propagation of this first higher-order mode. The mode $\hat{p}_{00}$ is the plane wave mode and will always propagate.

Evanescent waves do not propagate energy. They decay exponentionally with the distance along the duct. In the low frequency limit $\omega \ll (\omega_{mn})_c$ the pressure perturbation due to an evanescent mode will decay faster than $\exp\left(-\frac{m\pi}{h_1}x_1\right)$. For mode (1,0) a distance h_1 is sufficient for a decay by a factor $\exp(\pi) \approx 23$. All other higher-order modes will decay even faster.